2.2 Hodge theory

Let M be a closed oriented Riemannian manifold. For $x \in M$, the metric on M induces a metric on T_x^*M, thus a metric g^Λ on $\Lambda^k T_x^* M$ for $k \leq n := \dim M$. Explicitly, let $e^1, \cdots, e^n$ be an orthonormal basis of T^*M. Then $e_I \in \Lambda^k T^*M$ with $I = \{i_1 < \cdots < i_k\}$ forms an orthonormal basis of $\Lambda^k T_x^* M$. The volume form is locally given by

$$\mathrm{vol} := e^1 \wedge \cdots \wedge e^n. \tag{2.2.1}$$

Definition 2.2.1. The Hodge $*$-operator

$$* : \Lambda^k T^*M \to \Lambda T^{n-*}M \tag{2.2.2}$$

is defined by

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}, \tag{2.2.3}$$

for $\{i_1, \cdots, i_k, j_1, \cdots, j_{n-k}\} = \{1, \cdots, n\}$. In particular, we have $*1 = \mathrm{vol}$ and $* \,\mathrm{vol} = 1$.

Proposition 2.2.2. *(1) for any $\alpha, \beta \in \Lambda^k T^*M$, we have*

$$\alpha \wedge *\beta = g^\Lambda(\alpha, \beta) \,\mathrm{vol} \tag{2.2.4}$$

(2) On $\Lambda^k T^*M$,

$$*^2 = (-1)^{k(n-k)}. \tag{2.2.5}$$

(3) The $*$-operator is an isometry:

$$g^\Lambda(*\alpha, *\beta) = g^\Lambda(\alpha, \beta). \tag{2.2.6}$$

(4) For $\alpha \in \Lambda^k T^*M$, we have

$$g^\Lambda(\alpha, *\beta) = (-1)^{k(n-k)} g^\Lambda(*\alpha, \beta). \tag{2.2.7}$$

Proof. For (1), by (2.2.3),

$$\begin{aligned} e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge *(e^{i_1} \wedge \cdots \wedge e^{i_k}) \\ = \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} = \mathrm{vol}\,. \end{aligned} \tag{2.2.8}$$

For (2), by (2.2.3), we have

$$\begin{aligned} *^2\, (e^{i_1} \wedge \cdots \wedge e^{i_k}) &= \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} * e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} \\ &= \delta_{i_1,\cdots,i_k,j_1,\cdots,j_{n-k}} \delta_{j_1,\cdots,j_{n-k},i_1,\cdots,i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \\ &= (-1)^{k(n-k)} e^{i_1} \wedge \cdots \wedge e^{i_k}. \end{aligned} \tag{2.2.9}$$

For (3), by (2.2.5),

$$g^\Lambda(*\alpha, *\beta)\,\mathrm{vol} = *\alpha \wedge *^2\beta = (-1)^{k(n-k)} *\alpha \wedge \beta = \beta \wedge *\alpha$$
$$= g^\Lambda(\beta, \alpha)\,\mathrm{vol} = g^\Lambda(\alpha, \beta)\,\mathrm{vol} \quad (2.2.10)$$

For (4), by (2.2.5) and (2.2.6),

$$g^\Lambda(\alpha, *\beta) = g^\Lambda(*\alpha, *^2\beta) = (-1)^{k(n-k)} g^\Lambda(*\alpha, \beta) \quad (2.2.11)$$

The proof of our proposition is completed. □

Definition 2.2.3. We define an inner product on forms $\langle \cdot, \cdot \rangle_{\mathbb{R}} : \Omega^k(M, \mathbb{R}) \times \Omega^k(M, \mathbb{R}) \to \mathbb{R}$ by

$$\langle \alpha, \beta \rangle_{\mathbb{R}} := \int_M g^\Lambda(\alpha, \beta) dv = \int_M \alpha \wedge *\beta. \quad (2.2.12)$$

We denote by $d^* : \Omega^*(M, \mathbb{R}) \to \Omega^{*-1}(M, \mathbb{R})$ the formal adjoint of d with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, i.e., for any $\alpha, \beta \in \Omega^*(M, \mathbb{R})$,

$$\langle d\alpha, \beta \rangle_{\mathbb{R}} = \langle \alpha, d^*\beta \rangle_{\mathbb{R}}. \quad (2.2.13)$$

Proposition 2.2.4. *On $\Omega^k(M)$,*

$$d^* = (-1)^{n(k-1)+1} * d * . \quad (2.2.14)$$

Proof. By stokes' formula and Proposition 2.2.2, for $\alpha \in \Omega^{k-1}(M, \mathbb{R}), \beta \in \Omega^k(M, \mathbb{R})$, we have

$$\langle d\alpha, \beta \rangle_{\mathbb{R}} = \int_M d\alpha \wedge *\beta = -(-1)^{k-1} \int_M \alpha \wedge d * \beta$$
$$= (-1)^{k+(k-1)(n-k+1)} \int_M \alpha \wedge *^2 d * \beta = (-1)^{n(k-1)+1} \langle \alpha, *d * \beta \rangle_{\mathbb{R}}. \quad (2.2.15)$$

The proof of our proposition is completed. □

Since $d^2 = 0$, by (2.2.13), we have

$$(d^*)^2 = 0. \quad (2.2.16)$$

We define the Laplace-Beltrami operator $\Delta_{\mathbb{R}}$ by

$$\Delta_{\mathbb{R}} := (d + d^*)^2 = dd^* + d^*d. \quad (2.2.17)$$

Proposition 2.2.5. *We have*

$$\ker(\Delta_{\mathbb{R}}) = \ker(d) \cap \ker(d^*). \tag{2.2.18}$$

Proof. The proposition follows from

$$\langle \Delta_{\mathbb{R}}\alpha, \alpha\rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2. \tag{2.2.19}$$

The proof is completed. □

Theorem 2.2.6 (Hodge Theorem, real version)**.** *For any* $k \in \mathbb{N}$*, we have the orthogonal decomposition, called the Hodge decomposition*

$$\begin{aligned} \Omega^k(M) &= \ker(\Delta_{\mathbb{R}}|_{\Omega^k}) \oplus \mathrm{Im}(\Delta_{\mathbb{R}}|_{\Omega^k}) && (2.2.20)\\ &= \ker(\Delta_{\mathbb{R}}|_{\Omega^k}) \oplus \mathrm{Im}(d|_{\Omega^{k-1}}) \oplus \mathrm{Im}(d^*|_{\Omega^{k+1}}) && (2.2.21)\end{aligned}$$

and the canonical isomorphism

$$\ker(\Delta_{\mathbb{R}}|_{\Omega^k}) \simeq H^k(M,\mathbb{R}). \tag{2.2.22}$$

Especially, the space $\ker(\Delta_{\mathbb{R}}|_{\Omega^k})$ *is finite-dimensional.*

Corollary 2.2.7 (Poincaré duality)**.** *The bilinear form* $\int_M \alpha\wedge\beta$ *induces a non-degenerate pairing*

$$H^k(M,\mathbb{R}) \times H^{n-k}(M,\mathbb{R}) \to \mathbb{R}. \tag{2.2.23}$$

In other words, we get

$$H^k(M,\mathbb{R}) \simeq (H^{n-k}(M,\mathbb{R}))^*. \tag{2.2.24}$$

Proof. Take $[\alpha] \in H^k(M,\mathbb{R})$. Then by Hodge theorem, there exists $\alpha \in [\alpha]$ such that $\alpha \in \ker(\Delta_{\mathbb{R}}|_{\Omega^k})$. Thus by Proposition 2.2.5, $d^*\alpha = 0$. By Proposition 2.2.4, we have $d{*}\alpha = 0$. If $\int_M \alpha\wedge\beta = 0$ for any $\beta \in H^{n-k}(M,\mathbb{R})$, then $\int_M |\alpha|^2 dv = \int_M \alpha\wedge *\alpha = 0$. Thus $[\alpha] = 0$.

The proof of the corollary is completed. □

Now we assume that M is a closed complex manifold with $\dim_{\mathbb{C}} M = n$. As usual, let g be a Riemannian metric on TM. Then it could be $\mathbb{C}$-linearly extended on $TM \otimes \mathbb{C}$. We denote by

$$T^{*(p,q)}M = \Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M). \tag{2.2.25}$$

Then by (1.1.1)

$$\Lambda^k(T^*M\otimes\mathbb{C}) = \bigoplus_{p+q=k} T^{*(p,q)}M. \tag{2.2.26}$$

From (1.2.12), the Riemannian metric g on TM induces a Hermitian metric h on $T^{(1,0)}M$, thus a Hermitian metric h^Λ on $T^{*(p,q)}M$. As in (1.2.12), for $\alpha, \beta \in \Omega^{p,q}(M)$, we have

$$h^\Lambda(\alpha, \beta) = g^\Lambda(\alpha, \bar{\beta}). \tag{2.2.27}$$

We extend the Hodge $*$-operator $\mathbb{C}$-linearly to

$$* : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{2n-k}(T^*M \otimes \mathbb{C}). \tag{2.2.28}$$

By Definition 2.2.3, we have

$$* : T^{*(p,q)}M \to T^{*(n-q,n-p)}M. \tag{2.2.29}$$

As in Definition 2.2.3, we define the Hermitian inner product $\langle \cdot, \cdot \rangle : \Omega^{p,q}(M) \times \Omega^{p,q}(M) \to \mathbb{C}$ by

$$\langle \alpha, \beta \rangle_{\mathbb{C}} := \int_M h^\Lambda(\alpha, \beta) dv = \int_M \alpha \wedge *\bar{\beta}. \tag{2.2.30}$$

By Definition 2.2.3 and Proposition 2.2.4, since $\dim_{\mathbb{R}} M$ is even, we have the following proposition.

Proposition 2.2.8. *Let ∂^* and $\bar{\partial}^*$ be the formal adjoint of ∂ and $\bar{\partial}$ respectively. Then we have*

$$d^* = \partial^* + \bar{\partial}^*, \quad (\partial^*)^2 = (\bar{\partial}^*)^2 = 0. \tag{2.2.31}$$

and

$$\partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial * \tag{2.2.32}$$

Definition 2.2.9. The Laplacians associated with ∂ and $\bar{\partial}$ are defined as

$$\Delta_\partial = (\partial + \partial^*)^2 = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{2.2.33}$$

Clearly,

$$\Delta_\partial, \Delta_{\bar{\partial}} : \Omega^{p,q}(M) \to \Omega^{p,q}(M). \tag{2.2.34}$$

The following proposition is an analogue of Proposition 2.2.5. The proof is the same.

Proposition 2.2.10. *We have*

$$\ker(\Delta_\partial) = \ker(\partial) \cap \ker(\partial^*), \quad \ker(\Delta_{\bar{\partial}}) = \ker(\bar{\partial}) \cap \ker(\bar{\partial}^*). \tag{2.2.35}$$

Theorem 2.2.11 (Hodge Theorem, complex version)**.** *Let M be a closed complex manifold. Then we have two natural orthogonal decompositions*

$$\Omega^{p,q}(M) = \ker(\Delta_\partial|_{\Omega^{p,q}}) \oplus \mathrm{Im}(\partial|_{\Omega^{p-1,q}}) \oplus \mathrm{Im}(\partial^*|_{\Omega^{p+1,q}}) \tag{2.2.36}$$

and

$$\Omega^{p,q}(M) = \ker(\Delta_{\bar\partial}|_{\Omega^{p,q}}) \oplus \mathrm{Im}(\bar\partial|_{\Omega^{p,q-1}}) \oplus \mathrm{Im}(\bar\partial^*|_{\Omega^{p,q+1}}). \tag{2.2.37}$$

The spaces $\ker(\Delta_\partial|_{\Omega^{p,q}})$ *and* $\ker(\Delta_{\bar\partial}|_{\Omega^{p,q}})$ *are finite dimensional. And*

$$\ker(\Delta_{\bar\partial}|_{\Omega^{p,q}}) \simeq H^{p,q}(M), \tag{2.2.38}$$

the (p,q)*-Dolbeault cohomology.*

Let E be a holomorphic vector bundle over M. In Definition 2.1.28, the operator $\bar\partial^E$ induces the Dolbeault cohomology group $H^*(M,E)$. Let h^E be a Hermitian metric on E. As in Definition 2.2.3, we define an inner product on forms $\langle\cdot,\cdot\rangle_E : \Omega^{0,q}(M,E)\times\Omega^{0,q}(M,E)\to\mathbb{C}$ by

$$\langle s,t\rangle_E := \int_M h^{\Lambda\otimes E}(s,t)dv. \tag{2.2.39}$$

Here $h^{\Lambda\otimes E}$ denotes by the Hermitian metric on $\Lambda^*(T^*M\otimes\mathbb{C})\otimes E$ induced by h^Λ and h^E. We denote by $\bar\partial^{E,*} : \Omega^{0,*}(M,E)\to\Omega^{0,*-1}(M,E)$ the formal adjoint of $\bar\partial^E$ with respect to $\langle\cdot,\cdot\rangle_E$, i.e., for any $s,t\in\Omega^*(M,E)$,

$$\langle\bar\partial^E s,t\rangle_E = \langle s,\bar\partial^{E,*}t\rangle_E. \tag{2.2.40}$$

As in (2.2.16), we have

$$\left(\bar\partial^{E,*}\right)^2 = 0. \tag{2.2.41}$$

Definition 2.2.12. The Hermitian metric h^E on E induces a $\mathbb{C}$-anti-linear isomorphism $h : E\simeq E^*$. The map

$$\bar{*}_E : T^{*(p,q)}M\otimes E\to T^{*(n-p,n-q)}M\otimes E^* \tag{2.2.42}$$

is defined by $\bar{*}_E(\alpha\otimes A) = *(\bar\alpha)\otimes h^E(A)$.

With this notation, for $s,t\in T^{*(p,q)}M\otimes E$,

$$h^{\Lambda\otimes E}(s,t) = s\wedge\bar{*}_E(t), \tag{2.2.43}$$

where "Λ" is the exterior product in the form part and the evaluation map $E\otimes E^*\to\mathbb{C}$ in the bundle part. From Proposition 2.2.2 (2), on $T^{*(p,q)}M\otimes E$,

$$\bar{*}_{E^*}\circ\bar{*}_E = (-1)^{p+q}. \tag{2.2.44}$$

Proposition 2.2.13. *The formal adjoint operator*

$$\bar\partial^{E,*} = -\bar{*}_{E^*} \circ \bar\partial^{E^*} \circ \bar{*}_E. \tag{2.2.45}$$

Proof. For any holomorphic sections $s = \alpha \otimes A \in \Omega^{p,q}(M,E)$ and $t = \beta \otimes A' \in \Omega^{p,q+1}(M,E)$,

$$\begin{aligned}
\langle s, \bar\partial^{E,*} t\rangle_E = \langle \bar\partial^E s, t\rangle_E = \int_M \bar\partial^E s \wedge \bar{*}_E t = \int_M \bar\partial\alpha \wedge *\bar\beta \otimes A \otimes h(A') \\
\int_M (\bar\partial(\alpha \wedge *\bar\beta \otimes A \otimes h(A')) - (-1)^{p+q+1}\alpha \wedge \bar\partial(*\bar\beta \otimes A \otimes h(A'))) \\
= \int_M d(\alpha \wedge *\bar\beta \otimes A \otimes h(A')) - (-1)^{p+q+1} \int_M \alpha \wedge \bar\partial(*\bar\beta \otimes A \otimes h(A')) \\
= -(-1)^{p+q+1} \int_M s \wedge \bar\partial^{E^*}(\bar{*}_E t) = -\int_M s \wedge \bar{*}_{E^*} \circ \bar{*}_{E^*} \bar\partial^E (\bar{*}_E t) \\
= -\langle s, \bar{*}_{E^*} \circ \bar\partial^{E^*} \circ \bar{*}_E t\rangle_E.
\end{aligned} \tag{2.2.46}$$

The proof of our proposition is completed. ☐

Definition 2.2.14. The Laplacian associated with $\bar\partial^E$, which is called the Kodaira-Laplacian, is defined as

$$\Box^E = (\bar\partial^E + \bar\partial^{E,*})^2 = \bar\partial^E \bar\partial^{E,*} + \bar\partial^{E,*}\bar\partial^E = \left[\bar\partial^E, \bar\partial^{E,*}\right]. \tag{2.2.47}$$

Theorem 2.2.15 (Hodge Theorem, holomorphic bundle version)**.** *Let M be a closed complex manifold and E be a holomorphic vector bundle over M. Then we have the orthogonal decomposition*

$$\Omega^{0,q}(M,E) = \ker(\Box^E|_{\Omega^{0,q}}) \oplus \mathrm{Im}(\bar\partial^E|_{\Omega^{0,q-1}}) \oplus \mathrm{Im}(\bar\partial^{E,*}|_{\Omega^{0,q+1}}). \tag{2.2.48}$$

The spaces $\ker(\Box^E|_{\Omega^{0,q}})$ is finite dimensional. And

$$\ker(\Box^E|_{\Omega^{0,q}}) \simeq H^{0,q}(M,E). \tag{2.2.49}$$

Theorem 2.2.16 (Serre duality)**.** *Let M be a closed connected complex manifold. For $s \in \Omega^{0,q}(M,E)$, $t \in \Omega^{0,n-q}(M, K_M \otimes E^*) = \Omega^{n,n-q}(M,E^*)$, the bilinear form $\int_M s \wedge t$ induces a non-degenerate pairing*

$$H^q(M,E) \times H^{n-q}(M, K_M \otimes E^*) \to \mathbb{C}. \tag{2.2.50}$$

In other words, we get

$$H^q(M,E) \simeq (H^{n-q}(M, K_M \otimes E^*))^*. \tag{2.2.51}$$

Proof. Take $[\alpha] \in H^q(M,E)$. Then by Hodge theorem, there exists $\alpha \in [\alpha]$ such that $\alpha \in \ker(\square^E|_{\Omega^{0,q}})$. Thus by Proposition 2.2.13, we have $\bar{\partial}^{E^*}\bar{*}_E\alpha = 0$. If $\int_M \alpha \wedge \beta = 0$ for any $\beta \in H^{n-q}(M, K_M \otimes E^*)$, then $\int_M |\alpha|^2 dv = \int_M \alpha \wedge \bar{*}_E\alpha = 0$. Thus $[\alpha] = 0$.

The proof of the theorem is completed. □

By taking $E = T^{*(p,0)}M$, we have

Corollary 2.2.17 (Serre duality)**.** *Let M be a closed connected complex manifold. The bilinear form $\int_M \alpha \wedge \beta$ induces a non-degenerate pairing*

$$H^{p,q}(M) \times H^{n-p,n-q}(M) \to \mathbb{C}. \tag{2.2.52}$$

In other words, we get

$$H^{p,q}(M) \simeq (H^{n-p,n-q}(M))^*. \tag{2.2.53}$$

Remark that (2.2.51) is $\mathbb{C}$-linear and does not depend on the metrics on M and E.

Let ∇^E be the Chern connection on E. Recall that $(\nabla^E)^{1,0}$ is the $(1,0)$-part of ∇^E defined in (1.2.25). We denote by $(\nabla^E)^*$ and $(\nabla^E)^{1,0*}$ the formal adjoints of ∇^E and $(\nabla^E)^{1,0}$ with respect to $\langle\cdot,\cdot\rangle_E$ in (2.2.39) respectively.

Recall that $\widetilde{\nabla}$ is the connection defined in Proposition 1.2.14. It is a connection on $TM \otimes \mathbb{C}$ and it preserves TM. We still denote by $\widetilde{\nabla}$ the induced connection on TM. Then it preserves the metric on M. Let T be the torsion of $\overline{\nabla}$. Then $T \in \Lambda^2(T^*M) \otimes TM$ is defined by

$$T(U,V) = \widetilde{\nabla}_U V - \widetilde{\nabla}_V U - [U,V], \tag{2.2.54}$$

for vector fields U, V. Then T maps $T^{(1,0)}M \otimes T^{(1,0)}M$ (resp. $T^{(0,1)}M \otimes T^{(0,1)}M$) into $T^{(1,0)}M$ (resp. $T^{(1,0)}M$) and vanish on $T^{(1,0)}M \otimes T^{(0,1)}M$. Indeed, for $U = U_i\frac{\partial}{\partial z_i} \in T^{1,0}M$, $V = V_j\frac{\partial}{\partial \bar{z}_j} \in T^{0,1}M$, we have

$$\widetilde{\nabla}_V U = \nabla_V^{T^{(1,0)}M} U = i_V \bar{\partial}^{T^{(1,0)}M} U = V_j \frac{\partial U_i}{\partial \bar{z}_j}\frac{\partial}{\partial z_i}, \tag{2.2.55}$$

and

$$\widetilde{\nabla}_U V = \overline{\nabla_{\overline{U}}^{T^{(1,0)}M}\overline{V}} = U_i \frac{\partial V_j}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}. \tag{2.2.56}$$

Thus we have

$$\widetilde{\nabla}_U V - \widetilde{\nabla}_V U = [U,V]. \tag{2.2.57}$$

Let

$$\widetilde{\nabla}^E = \widetilde{\nabla} \otimes 1 + 1 \otimes \nabla^E. \tag{2.2.58}$$

Lemma 2.2.18. *Let $\{e_j\}$ be a locally orthonormal basis of TM and $\{e^j\}$ be the duals. We have*

$$\nabla^E = e^j \wedge \widetilde{\nabla}^E_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^j \wedge e^k i_{e_l}, \tag{2.2.59}$$

$$(\nabla^E)^* = -i_{e_j} \wedge \widetilde{\nabla}^E_{e_j} - g(T(e_j, e_k), e_k) i_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^l \wedge i_{e_k} i_{e_j}. \tag{2.2.60}$$

Especially, if $E = \mathbb{C}$, we have

$$d = e^j \wedge \widetilde{\nabla}_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^j \wedge e^k i_{e_l}, \tag{2.2.61}$$

$$d^* = -i_{e_j} \wedge \widetilde{\nabla}_{e_j} - g(T(e_j, e_k), e_k) i_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^l \wedge i_{e_k} i_{e_j}. \tag{2.2.62}$$

Proof. We prove (2.2.61) first. We denote by $\mathbf{d}$ the right hand side of (2.2.61). It is easy to see that for any homogeneous differential forms α, β, we have

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathbf{d}\beta. \tag{2.2.63}$$

So we only need to show that $\mathbf{d}$ agrees with d on functions, which is clear, and 1-forms. For any $f \in \mathscr{C}^\infty(M)$,

$$\begin{aligned} e^j \wedge \widetilde{\nabla}_{e_j} df &= e^j \wedge e^k \langle \widetilde{\nabla}_{e_j} df, e_k \rangle = e^j \wedge e^k \big(e_j(e_k(f)) - \langle df, \widetilde{\nabla}_{e_j} e_k \rangle \big) \\ &= \frac{1}{2} e^j \wedge e^k \big(e_j(e_k(f)) - \langle df, \widetilde{\nabla}_{e_j} e_k \rangle - e_k(e_j(f)) - \langle df, \widetilde{\nabla}_{e_k} e_j \rangle \big) \\ &= -\frac{1}{2} e^j \wedge e^k \langle T(e_j, e_k), df \rangle. \end{aligned} \tag{2.2.64}$$

Thus $\mathbf{d}$ coincides with d on 1-forms. Thus we get (2.2.61).

For (2.2.59), let $s = \alpha \otimes A \in \Omega^*(M, E)$. Then by (2.2.61),

$$\begin{aligned} \widetilde{\nabla}^E(\alpha \otimes A) &= d\alpha \otimes A + (-1)^{\deg \alpha} \alpha \wedge \nabla^E A \\ &= e^j \wedge \widetilde{\nabla}^E_{e_j} + \frac{1}{2} g(T(e_j, e_k), e_l) e^j \wedge e^k i_{e_l}. \end{aligned} \tag{2.2.65}$$

Now we prove (2.2.62). From the knowledge of differential geometry, for any $\theta \in \Omega^1(M)$, the function $\operatorname{tr}(\nabla\theta)$ is given by

$$\operatorname{tr}(\nabla\theta) = e_j(\alpha(e_j)) - \theta(\nabla_{e_j} e_j). \tag{2.2.66}$$

Then we have

$$\int_M \mathrm{tr}(\nabla\theta)dv = 0. \tag{2.2.67}$$

For $\alpha, \beta \in \Omega^*(M)$, take $\theta = -g^\Lambda(i.\alpha, \beta)$. We have

$$\mathrm{tr}(\nabla\alpha) = -e_j\big(g^\Lambda(i_{e_j}\alpha, \beta)\big) + g^\Lambda(i_{\nabla_{e_j}e_j}\alpha, \beta). \tag{2.2.68}$$

Since $i_{e_j}\widetilde{\nabla}_{e_j}\alpha = \widetilde{\nabla}_{e_j}i_{e_j}\alpha - i_{\widetilde{\nabla}_{e_j}e_j}\alpha$, we have

$$\begin{aligned} g^\Lambda(e^j \wedge \widetilde{\nabla}_{e_j}\alpha, \beta) = g^\Lambda(\widetilde{\nabla}_{e_j}\alpha, i_{e_j}\beta) &= e_j\big(g^\Lambda(\alpha, i_{e_j}\beta)\big) - g^\Lambda(\alpha, \widetilde{\nabla}_{e_j}i_{e_j}\beta) \\ &= e_j\big(g^\Lambda(\alpha, i_{e_j}\beta)\big) - g^\Lambda(\alpha, i_{e_j}\widetilde{\nabla}_{e_j}\beta) + g^\Lambda(\alpha, i_{\widetilde{\nabla}_{e_j}e_j}\beta) \\ &= -g^\Lambda(\alpha, i_{e_j}\widetilde{\nabla}_{e_j}\beta) - \mathrm{tr}(\nabla\theta) - g(T(e_k, e_j), e_j)g^\Lambda(\alpha, i_{e_k}\beta) \end{aligned} \tag{2.2.69}$$

Thus

$$\big(e^j \wedge \widetilde{\nabla}_{e_j}\big)^* = -i_{e_j}\widetilde{\nabla}_{e_j} - g(T(e_j, e_k), e_k)i_{e_j}. \tag{2.2.70}$$

We get (2.2.62).

Using the same argument in (2.2.65), we get (2.2.60).

The proof of our lemma is completed. □

Let $\mathcal{A}$ be a ring and $f, g : TM \otimes T^*M \otimes \mathbb{C} \to \mathcal{A}$ be two linear maps. Then from (1.1.18), we have

$$\begin{aligned} \sum_{i=1}^{2n} f(e^i)g(e_i) &= \sum_{i=1}^{n} \big(f(\theta^i)g(\theta_i) + f(\bar\theta^i)g(\bar\theta_i)\big), \\ \sum_{i=1}^{2n} f(e_i)g(e_i) &= \sum_{i=1}^{n} \big(f(\theta_i)g(\bar\theta_i) + f(\bar\theta_i)g(\theta_i)\big). \end{aligned} \tag{2.2.71}$$

By taking the $(1,0)$-part and the $(0,1)$-part of (2.2.59) and (2.2.60) and using (2.2.71), we have the following lemma.

Lemma 2.2.19. *Let $\{\theta_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}M$. Then we have*

$$\bar\partial^E = \bar\theta^j \wedge \widetilde{\nabla}^E_{\bar\theta_j} + \frac{1}{2}g(T(\bar\theta_j, \bar\theta_k), \theta_l)\bar\theta^j \wedge \bar\theta^k i_{\bar\theta_l}, \tag{2.2.72}$$

$$\bar\partial^{E,*} = -i_{\bar\theta_j} \wedge \widetilde{\nabla}^E_{\theta_j} - g(T(\theta_j, \theta_k), \bar\theta_k)i_{\bar\theta_j} + \frac{1}{2}g(T(\theta_j, \theta_k), \bar\theta_l)\bar\theta^l \wedge i_{\bar\theta_k}i_{\bar\theta_j}, \tag{2.2.73}$$

$$(\nabla^E)^{1,0} = \theta^j \wedge \widetilde{\nabla}^E_{\theta_j} + \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \theta^k i_{\theta_l}, \tag{2.2.74}$$

$$(\nabla^E)^* = -i_{\theta_j} \wedge \widetilde{\nabla}^E_{\bar{\theta}_j} - g(T(\bar{\theta}_j, \bar{\theta}_k), \theta_k) i_{\theta_j} + \frac{1}{2} g(T(\bar{\theta}_j, \bar{\theta}_k), \theta_l)\theta^l \wedge i_{\theta_k} i_{\theta_j}. \tag{2.2.75}$$

Let ω be the real $(1,1)$-form associated with g in (1.1.13).

Definition 2.2.20. We define the **Lefschetz operator** $L = (\omega\wedge) \otimes 1$ on $\Lambda^{\cdot,\cdot}(T^*M) \otimes E$ and its adjoint $\Lambda = i(\omega)$ with respect to $h^{\Lambda \otimes E}$.

For $\{\theta_j\}_{j=1}^n$ a local orthonormal frame of $T^{(1,0)}M$, by (1.1.13), we have

$$L = \sqrt{-1}\theta^j \wedge \bar{\theta}^j \wedge, \quad \Lambda = -\sqrt{-1} i_{\bar{\theta}_j} i_{\theta_j}. \tag{2.2.76}$$

It is easy to see that

$$\Lambda = *^{-1} \circ L \circ *. \tag{2.2.77}$$

Definition 2.2.21. The holomorphic Kodaira Laplacian is defined by

$$\overline{\square}^E = \left[(\nabla^E)^{1,0}, (\nabla^E)^{1,0*}\right] = (\nabla^E)^{1,0}(\nabla^E)^{1,0*} + (\nabla^E)^{1,0*}(\nabla^E)^{1,0}. \tag{2.2.78}$$

The Hermitian torsion operator is defined by

$$\mathcal{T} := [\Lambda, \partial\omega] = [i(\omega), \partial\omega]. \tag{2.2.79}$$

Theorem 2.2.22 (Generalized Kähler identities)**.**

$$\left[\bar{\partial}^{E,*}, L\right] = \sqrt{-1}\left((\nabla^E)^{1,0} + \mathcal{T}\right), \tag{2.2.80}$$

$$\left[(\nabla^E)^{1,0*}, L\right] = -\sqrt{-1}\left(\bar{\partial}^E + \overline{\mathcal{T}}\right), \tag{2.2.81}$$

$$\left[\Lambda, \bar{\partial}^E\right] = -\sqrt{-1}\left((\nabla^E)^{1,0*} + \mathcal{T}^*\right), \tag{2.2.82}$$

$$\left[\Lambda, (\nabla^E)^{1,0}\right] = \sqrt{-1}\left(\bar{\partial}^{E,*} + \overline{\mathcal{T}}^*\right), \tag{2.2.83}$$

$$\left[\bar{\partial}^E, L\right] = \left[(\nabla^E)^{1,0}, L\right] = \left[\Lambda, \bar{\partial}^{E,*}\right] = \left[\Lambda, (\nabla^E)^{1,0*}\right] = 0. \tag{2.2.84}$$

Proof. Note that (2.2.82) and (2.2.83) are the adjoints of (2.2.80) and (2.2.81). We only need to prove the first two formulas.

From (2.2.73),

$$\left[\bar{\partial}^{E,*}, L\right] = -\left[i_{\bar{\theta}_j} \wedge \widetilde{\nabla}^E_{\theta_j}, L\right] - g(T(\theta_j, \theta_k), \bar{\theta}_k)\left[i_{\bar{\theta}_j}, L\right] + \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\left[\bar{\theta}^l \wedge i_{\bar{\theta}_k} i_{\bar{\theta}_j}, L\right]. \quad (2.2.85)$$

By (2.2.76),

$$\left[i_{\bar{\theta}_j}, L\right] = -\sqrt{-1}\theta^j \wedge . \quad (2.2.86)$$

Also by (2.2.76),

$$\begin{aligned}[\widetilde{\nabla}^E_{\theta_j}, L] &= \sqrt{-1}(\widetilde{\nabla}_{\theta_j}\theta^k) \wedge \bar{\theta}^k \wedge + \sqrt{-1}\theta^k \wedge (\widetilde{\nabla}_{\theta_j}\bar{\theta}^k)\wedge \\ &= \sqrt{-1}\big(- g(\widetilde{\nabla}_{\theta_j}\theta_l, \bar{\theta}_k) - g(\theta_l, \widetilde{\nabla}_{\theta_j}\bar{\theta}_k)\big)\theta^l \wedge \bar{\theta}^k \wedge = 0. \quad (2.2.87)\end{aligned}$$

Thus by (2.2.86) and (2.2.87),

$$-\left[i_{\bar{\theta}_j} \wedge \widetilde{\nabla}^E_{\theta_j}, L\right] = -\left[i_{\bar{\theta}_j}, L\right] \wedge \widetilde{\nabla}^E_{\theta_j} = \sqrt{-1}\theta^j \wedge \widetilde{\nabla}^E_{\theta_j}. \quad (2.2.88)$$

From (2.2.86), we have

$$\begin{aligned}\left[\bar{\theta}^l \wedge i_{\bar{\theta}_k} i_{\bar{\theta}_j}, L\right] &= \bar{\theta}^l \wedge \left(\left[i_{\bar{\theta}_k}, L\right] i_{\bar{\theta}_j} + i_{\bar{\theta}_k}\left[i_{\bar{\theta}_j}, L\right]\right) \\ &= -\sqrt{-1}\bar{\theta}^l \left(\theta^k \wedge i_{\bar{\theta}_j} + i_{\bar{\theta}_k}\theta^j\right). \quad (2.2.89)\end{aligned}$$

Thus,

$$\begin{aligned}\left[\bar{\partial}^{E,*}, L\right] &= \sqrt{-1}\theta^j \wedge \widetilde{\nabla}^E_{\theta_j} + \sqrt{-1}g(T(\theta_j, \theta_k), \bar{\theta}_k)\theta^j \\ &\quad + \sqrt{-1}g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \bar{\theta}^l \wedge i_{\theta_j}. \quad (2.2.90)\end{aligned}$$

From (2.2.87), we see that $\widetilde{\nabla}\omega = 0$. By (2.2.74), we have

$$\partial\omega = \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \theta^k i_{\theta_l}\omega = \frac{\sqrt{-1}}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\theta^j \wedge \theta^k \wedge \bar{\theta}^l. \quad (2.2.91)$$

So from (2.2.90), since $[\Lambda, \theta^j] = -\sqrt{-1}i_{\bar{\theta}_j}$ and $[\Lambda, \bar{\theta}^j] = -\sqrt{-1}i_{\theta_j}$ we have

$$\mathcal{T} = \frac{1}{2} g(T(\theta_j, \theta_k), \bar{\theta}_l)\left(2\theta^k \wedge \bar{\theta}^l \wedge i_{\bar{\theta}_j} - 2\delta_{jl}\theta^k - \theta^j \wedge \theta^k \wedge i_{\theta_l}\right). \quad (2.2.92)$$

By (2.2.74), (2.2.90) and (2.2.92), we get (2.2.80).

As the computation is local, we can choose a locally holomorphic frame of E to reduce the proof of (2.2.81) to the case that E is a trivial bundle. Then (2.2.81) follows from (2.2.80) by conjugation.

The formula (2.2.84) follows directly from the Leibniz's rule.

The proofs of the generalized Kähler identities are completed. □

For super-commutator

$$[B, C] = BC - (-1)^{|B||C|} CB, \tag{2.2.93}$$

where $|\cdot|$ is the degree, the Jacobi identity reads

$$(-1)^{|C||A|}[A, [B, C]] + (-1)^{|A||B|}[B, [C, A]] + (-1)^{|B||C|}[C, [A, B]] = 0. \tag{2.2.94}$$

Theorem 2.2.23 (Bochner-Kodaira-Nakano formula)**.**

$$\square^E = \overline{\square}^E + \sqrt{-1}[R^E, \Lambda] + [(\nabla^E)^{1,0}, \mathcal{T}^*] - [\bar{\partial}^E, \overline{\mathcal{T}}^*]. \tag{2.2.95}$$

Proof. From Theorem 2.2.22, (1.2.28), (2.2.47), (2.2.78) and (2.2.94), we have

$$\begin{aligned} \square^E = \left[\bar{\partial}^E, \bar{\partial}^{E,*}\right] &= -\sqrt{-1}\left[\bar{\partial}^E, \left[\Lambda, (\nabla^E)^{1,0}\right]\right] - \left[\bar{\partial}^E, \overline{\mathcal{T}}^*\right] \\ &= -\sqrt{-1}\left[\Lambda, \left[(\nabla^E)^{1,0}, \bar{\partial}^E\right]\right] - \sqrt{-1}\left[(\nabla^E)^{1,0}, \left[\bar{\partial}^E, \Lambda\right]\right] - \left[\bar{\partial}^E, \overline{\mathcal{T}}^*\right] \\ &= -\sqrt{-1}\left[\Lambda, R^E\right] + \left[(\nabla^E)^{1,0}, (\nabla^E)^{1,0*}\right] + [(\nabla^E)^{1,0}, \mathcal{T}^*] - \left[\bar{\partial}^E, \overline{\mathcal{T}}^*\right]. \end{aligned} \tag{2.2.96}$$

The proof of our theorem is complete. □

Now we assume that (M, ω) is Kähler.

Theorem 2.2.24. *Assume that (M, ω) is Kähler. Then*

$$\begin{aligned} &[\bar{\partial}^*, L] = \sqrt{-1}\partial, \quad [\partial^*, L] = -\sqrt{-1}\bar{\partial}, \quad [\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*, \\ &[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*, \quad \left[\bar{\partial}, L\right] = [\partial, L] = \left[\Lambda, \bar{\partial}^*\right] = [\Lambda, \partial^*] = 0, \\ &\square^E = \overline{\square}^E + \sqrt{-1}[R^E, \Lambda], \quad \Delta_{\mathbb{R}} = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}. \end{aligned} \tag{2.2.97}$$

Proof. By Proposition 1.2.14 and (2.2.12), if (M, ω) is Kähler, $\mathcal{T} = 0$. Thus we only need to prove the last formula.

By Theorem 2.2.22, we have

$$[\partial, \bar{\partial}^*] = -\sqrt{-1}[\partial, [\Lambda, \partial]] = \partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial = 0. \tag{2.2.98}$$

Thus

$$\Delta_{\mathbb{R}} = -[d, d^*] = [\partial + \bar{\partial}, \partial^* + \bar{\partial}^*] = \Delta_\partial + \Delta_{\bar{\partial}} + [\partial, \bar{\partial}^*] + [\bar{\partial}, \partial^*] \\ = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \quad (2.2.99)$$

The proof of our theorem is completed. □

The following theorem is the direct consequence of Theorem 2.2.22 and 2.2.24.

Theorem 2.2.25. *Assume that (M, ω) is Kähler. We denote by $\Delta := \Delta_\partial = 2Delta_{\bar{\partial}}$. Then*
(1) $H^k(M, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M)$;
(2) $H^{p,q}(M) \simeq \overline{H^{q,p}(M)}$ and Serre duality yields $H^{p,q}(M) \simeq H^{n-p,n-q}(M)^*$;
(3) Δ commutes with $*, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L$ and Λ.

Since $\Delta \circ * = * \circ \Delta$ and $*^2 = (-1)^{p(n-q)}$, the Hodge $*$-map induces an isomorphism

$$* : H^{p,q}(M) \simeq H^{n-q,n-p}(M). \quad (2.2.100)$$

Theorem 2.2.26 ($\partial\bar{\partial}$-lemma)**.** *Let M be a compact Kähler manifold. Then for a d-closed form α of type (p, q), the following conditions are equivalent:*
(1) The form α is d-exact, i.e., $\alpha = d\beta$ for some $\beta \in \Omega^{p+q+1}(M, \mathbb{C})$.
(2) The form α is ∂-exact, i.e., $\alpha = \partial\beta$ for some $\beta \in \Omega^{p-1,q}(M)$.
(3) The form α is $\bar{\partial}$-exact, i.e., $\alpha = \bar{\partial}\beta$ for some $\beta \in \Omega^{p,q-1}(M)$.
(4) The form α is $\partial\bar{\partial}$-exact, i.e., $\alpha = \partial\bar{\partial}\beta$ for some $\beta \in \Omega^{p-1,q-1}(M)$.

Proof. It is obvious that (4) implies (1), (2) and (3). By Hodge theory, if any of (1), (2) and (3) holds, we see that α is orthogonal to $\ker(\Delta)$. Since α is d-closed, it is ∂-closed and $\bar{\partial}$-closed. Since $\alpha \perp \mathrm{Im}\partial^*$, we have $\alpha = \partial\gamma$. Now we use the Hodge decomposition with respect to $\bar{\partial}$ to the form γ. Then $\gamma = \bar{\partial}\beta + \bar{\partial}^*\beta' + \beta''$ for some harmonic form β''. Thus, $\alpha = \partial\bar{\partial}\beta + \partial\bar{\partial}^*\beta'$. By (2.2.98), we have $\partial\bar{\partial}^* = -\bar{\partial}^*\partial$. Thus

$$\|\partial\bar{\partial}^*\beta'\|^2 = \|\bar{\partial}^*\partial\beta'\|^2 = (\bar{\partial}\bar{\partial}^*\partial\beta', \partial\beta') = (\bar{\partial}\partial\bar{\partial}\beta - \bar{\partial}\alpha, \partial\beta') = 0. \quad (2.2.101)$$

We have $\alpha = \partial\bar{\partial}\beta$.
The proof is completed. □